The Behavior of the Free Boundary for Reaction-Diffusion Equations with Convection in an Exterior Domain with Neumann or Dirichlet Boundary Condition

Abstract

Let

$$\mathcal{L} = A(r)\frac{d^2}{dr^2} - B(r)\frac{d}{dr}$$

be a second order elliptic operator and consider the reaction-diffusion equation with Neumann boundary condition,

$$\mathcal{L}u = \Lambda u^p \text{ for } r \in (R, \infty);$$
$$u'(R) = -h;$$
$$u \geq 0 \text{ is minimal,}$$

where $p \in (0,1)$, $R > 0$, $h > 0$ and $\Lambda = \Lambda(r) > 0$. This equation is the radially symmetric case of an equation of the form

$$Lu = \Lambda u^p \text{ in } \mathbb{R}^d - D;$$
$$\nabla u \cdot \bar{n} = -h \text{ on } \partial D;$$
$$u \geq 0 \text{ is minimal,}$$

where

$$L = \sum_{i,j=1}^{d} a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}$$

is a second order elliptic operator, and where $d \geq 2$, $h > 0$ is continuous, $D \subset R^d$ is bounded, and $\bar{n}$ is the unit inward normal to the domain $\mathbb{R}^d - \bar{D}$. Consider also the same equations with the Neumann boundary condition replaced by the Dirichlet boundary condition; namely, $u(R) = h$ in the radial case and $u = h$ on ∂D in the general case. The solutions to the above equations may possess a free boundary. In the radially symmetric case, if $r^*(h) = \inf\{r > R : u(r) = 0\} < \infty$, we call this the radius of the free boundary; otherwise there is no free boundary. We normalize the diffusion coefficient A to be on unit order, consider the convection vector field B to be on order r^m, $m \in R$, pointing either inward $(-)$ or outward $(+)$, and consider the reaction coefficient Λ to be on order r^{-j}, $j \in R$. For both the Neumann boundary case and the Dirichlet boundary case, we show for which choices of m, $(\pm)$ and j a free boundary exists, and when it exists, we obtain its growth rate in h as a function of m, $(\pm)$ and j. These results are then used to study the free boundary in the non-radially symmetric case.